Representations of solvable Lie algebras with filtrations

A.N.Panov *

1 Introduction. Main definitions

This paper is devoted to classification of irreducible representations of Lie algebras. Complete classification is known only for Lie algebras of small dimension [1, 2]. This classification is not constructible and cannot be generalized to more complicated Lie algebras. A more reasonable approach is to classify not irreducible representations but their kernels (primitive ideals) in the universal enveloping algebra. Started from [3] there were many works in this direction.

An other possibility is to study some category of representations and irreducible representation. For instance, the category of representations of highest weight for semisimple Lie algebras. Irreducible representations in this category are irreducible factors of Verma modules [3, 4].

In the case of solvable Lie algebras one well known a family of irreducible representations $M_s(f)$, induced from the Vergne polarization [5]. In this paper we consider the category of representations of Lie algebras with fixed filtrations. We prove that every irreducible representation in this category have the form $M_s(f)$ for some $f \in \mathfrak{g}^*$ (theorem 2.4). That is the mapping $M_s : f \mapsto M_s(f)$ from $\mathfrak{g}^*$ to $\mathrm{Irr}(\mathfrak{g}, s)$ is surjective. The mapping M_s is extended to bijection of the factor set $\mathfrak{g}^*/\mathfrak{R}$ to $\mathrm{Irr}(\mathfrak{g}, s)$ (theorem 3.4 and definition 3.5). Remark that two representations of the form $M_s(f)$ are equivalent in the category of representations of Lie algebras with filtrations (i.e. $\mathfrak{R}$-equivalent) if and only if they are equivalent in the usual sense (theorem 3.4). The classes of $\mathfrak{R}$-equivalence are described in Theorem 3.7.

Using classes of $\mathfrak{R}$-equivalence on $\mathfrak{g}^*$ we find spectra of the representation induced by irreducible representation of subalgebra (theorem 4.3), spectra of restrictions of irreducible representation on subalgebra, (theorem 4.4) and spectra of the tensor product of two irreducible representations (theorem 4.5).

In the last section the connection between the map $M_s : \mathfrak{g}^*/\mathfrak{R} \to \mathrm{Irr}(\mathfrak{g}, s)$ and the Dixmier mapping is investigated. The constructed theory is an analog

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A.A.Kirillov's orbit method (see [3, 6, 7]) in the category of Lie algebras with filtration.

In 1979 this paper was deposed in All-Union Institute for Science and Technical Information (VINITY), see [8]. Turn to formulation of main definitions of the paper.

1.1. Let field K be a field of zero characteristic and $\mathfrak{g}$ be a solvable Lie algebra over K. A filtration $s_{\mathfrak{g}}$ on $\mathfrak{g}$ is a chain of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_{k-1} \supset \mathfrak{g}_k, \tag{1}$$

such that $\mathfrak{g}_k = \{0\}$ and codimension of $\mathfrak{g}_i$ in $\mathfrak{g}_{i-1}$ as less or equal to 1. A Lie algebra with filtration is a Lie algebra $\mathfrak{g}$ with the its fixed filtration $s_{\mathfrak{g}}$. To simplify notations we denote filtration by s and a Lie algebra $\mathfrak{g}$ with filtration s by $(\mathfrak{g}, s)$.

Remark. 1) A filtration on $\mathfrak{g}$ always exists, since any finite dimensional (in particular, adjoint) representation of $\mathfrak{g}$ can be performed in triangular form [3, 1.3.12].

2) In this definition we allow that some ideal may appear several times. This makes it possible to restrict a filtration on a subalgebra (see 1.6) and to consider a restriction of a representations of a Lie algebra with filtration on a subalgebra.

We say that two filtrations on a Lie algebra are equal, if they consists of the same ideals and may differ only by multiplicity of their entrance in filtration.

Let $(\mathfrak{g}', s')$ and $(\mathfrak{g}'', s'')$ be two Lie algebras with filtrations. A homomorphism $\phi : (\mathfrak{g}', s') \to (\mathfrak{g}'', s'')$ of Lie algebras with filtrations is a homomorphism $\phi : \mathfrak{g}' \to \mathfrak{g}''$ such that $\phi(\mathfrak{g}'_i) \subset \mathfrak{g}''_i$.

1.2. A chain of ideals $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k$ produces the chain of subalgebras $U(\mathfrak{g}_0) \supset U(\mathfrak{g}_1) \supset \ldots \supset U(\mathfrak{g}_k)$ of the universal enveloping algebra $U(\mathfrak{g})$. Here we put $U(\mathfrak{g}_k) = K$, since $\mathfrak{g}_k = \{0\}$. We denote by $U(\mathfrak{g}, s)$ the universal enveloping algebra $U(\mathfrak{g})$ with the fixed filtration.

1.3. Let V be a linear space over a field K (infinite dimensional, in general). A filtration s_V in V is a chain of embedded subspaces

$$V = V_0 \supset V_1 \supset \ldots \supset V_k.$$

To simplify notations we denote a filtration by s and a Lie algebra with filtration s by (V, s).

Remark. Note here that in these case there is no restrictions on the codimensions of V_i in V_{i-1} and on dimensions of V_k.

Assume that for every i we are given a representation ρ_i of the Lie algebra $\mathfrak{g}_i$ in linear space V_i, such that the subspace V_{i+1} is invariant with respect to

the restriction of ρ_i on $\mathfrak{g}_{i+1}$. Then we say that this is a representation (ρ, s) of the Lie algebra with filtration $(\mathfrak{g}, s)$ or that (V, s) is a $U(\mathfrak{g}, s)$-module.

We say that two representations (ρ', s) and (ρ'', s) of Lie algebras $(\mathfrak{g}, s)$ in the spaces (V, s) and (W, s) are equivalent, if there exists an isomorphism $C : V \to W$ such that $C(V_i) = W_i$ and $C\rho'(x) = \rho''(x)C$ for every $x \in \mathfrak{g}$.

1.4. A representation (ρ, s) is irreducible, if every ρ_i is an irreducible representation of $\mathfrak{g}_i$ in V_i. We denote the set of equivalence classes of irreducible representations of $(\mathfrak{g}, s)$ by $\mathrm{Irr}(\mathfrak{g}, s)$.

1.5. If (V, s) is a linear subspace with filtration and W is a subspace in V, then the filtration of V induces a filtration of W:

$$W = W_0 \supset W_1 \supset \ldots \supset W_k,$$

where $W_i = V_i \cap W$. The factor space V/W is also equipped with a natural filtration

$$(V_0 + W)/W \supset (V_1 + W)/W \supset \ldots \supset (V_k + W)/W,$$

that coincides with

$$V_0/W_0 \supset V_1/W_1 \supset \ldots \supset V_k/W_k.$$

If (V, s) is a $U(\mathfrak{g}, s)$-module and W is a $U(\mathfrak{g})$-submodule of V, then we say that (W, s) is a $U(\mathfrak{g}, s)$-submodule of (V, s).

A module T_s over $U(\mathfrak{g}, s)$ is called a subfactor module of (V, s), if there exists two $U(\mathfrak{g}, s)$-submodules (W, s) and (W', s) such that $W \supset W'$ and factor of (W, s) by (W', s) is isomorphic to T_s. The set of all irreducible subfactor modules of (V, s) is called the spectrum of (V, s) and is denoted by $\mathrm{Spec}(V, s)$.

1.6. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then filtration of $\mathfrak{g}$ produces a filtration

$$\mathfrak{g}_0 \cap \mathfrak{h} \supset \mathfrak{g}_1 \cap \mathfrak{h} \supset \ldots \supset \mathfrak{g}_k \cap \mathfrak{h}$$

of $\mathfrak{h}$. We shall also denote it by s.

If (ρ, s) is a representation of $(\mathfrak{g}, s)$, then the restrictions of ρ_i on $\mathfrak{h}_i = \mathfrak{g}_i \cap \mathfrak{h}$ in V_i defines a representation of $(\mathfrak{h}, s)$ in (V, s).

1.7. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and let (W, s) be an arbitrary $U(\mathfrak{h}, s)$-module. Recall that the induced module $\mathrm{ind}(W, \mathfrak{g})$ (see [3, 5.1]) is defined as linear space

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} W$$

with the left action of $U(\mathfrak{g})$. We consider a chain of subspaces in $\mathrm{ind}(W, \mathfrak{g})$

$$\mathrm{ind}(W_0, \mathfrak{g}) \supset \mathrm{ind}(W_1, \mathfrak{g}_1) \supset \ldots \supset \mathrm{ind}(W_k, \mathfrak{g}_k).$$

Since every $\mathrm{ind}(W_i, \mathfrak{g}_i)$ is a $U(\mathfrak{g}_i)$-module, $\mathrm{ind}(W, \mathfrak{g}, s)$ is a $U(\mathfrak{g}, s)$-module.

1.8. Let us define the tensor product of representations of Lie algebras with filtrations. Let $\mathfrak{g}$ be a Lie algebra with filtration s given as $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k$. We denote the direct sum of two copies of $\mathfrak{g}$ by $\mathfrak{g} \times \mathfrak{g}$ (instead of the usual notation $\mathfrak{g} \oplus \mathfrak{g}$). The Lie algebra $\mathfrak{g} \times \mathfrak{g}$ has a filtration

$$\mathfrak{g}_0 \times \mathfrak{g}_0 \supset \mathfrak{g}_0 \times \mathfrak{g}_1 \supset \mathfrak{g}_1 \times \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k \times \mathfrak{g}_k,$$

denoted by $s \times s$.

If V, W are $U(\mathfrak{g})$-modules, then we denote by $V \times W$ the linear space $V \otimes W$ considered as a $U(\mathfrak{g} \times \mathfrak{g})$-module.

If (V, s) and (W, s) are $U(\mathfrak{g}, s)$-modules, then $V \times W$ is a $U(\mathfrak{g} \times \mathfrak{g}, s \times s)$-module with filtration

$$V_0 \times W_0 \supset V_0 \times W_1 \supset V_1 \times W_1 \supset \ldots \supset V_k \times W_k.$$

The Lie algebra $\mathfrak{g}$ embedded into $\mathfrak{g} \times \mathfrak{g}$ diagonally and the filtration $s \times s$ induces the filtration $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_k$ on $\mathfrak{g}$ which coincides with s. The restriction of representation $V \times W$ to $(\mathfrak{g}, s)$ turns $V \times W$ into a $U(\mathfrak{g}, s)$-module, which we denote by $(V, s) \otimes (W, s)$.

1.9. Let $\mathfrak{g}$ be an arbitrary Lie algebra, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ and $f \in \mathfrak{g}^*$. A subalgebra $\mathfrak{p}$ is called a polarization of f, if $\mathfrak{p}$ is a maximal isotropic subspace with respect to a skew symmetric form $f([x, y])$, that is, $f([\mathfrak{p}, \mathfrak{p}]) = 0$ and $\dim \mathfrak{p} = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^f)$, where $\mathfrak{g}^f$ is the stabilizer of f in $\mathfrak{g}$. The restriction of f to its polarization is a one-dimensional representation (character) of the polarization.

Let $(\mathfrak{g}, s)$ be a Lie algebra with filtration. For any element $f \in \mathfrak{g}^*$ one can construct the polarization, that is called the Vergne polarization, by the following formula

$$\mathfrak{pv}(f) = \sum_{i=0}^{k} \mathfrak{g}_i^{f_i},$$

where f_i is a restriction of f to $\mathfrak{g}_i^*$, and $\mathfrak{g}_i^{f_i}$ is the stabilizer of f_i in $\mathfrak{g}_i$ (see [5], [3, 1.12.10]).

Denote by $M(f)$ the module over $U(\mathfrak{g})$, induced by the restriction of f to $\mathfrak{pv}(f)$. The module $M(f)$ is irreducible [3, 6.1.1].

The intersection $\mathfrak{pv}(f) \cap \mathfrak{g}_i$ coincides with the Vergne polarization $\mathfrak{pv}_i(f_i)$ of f_i in $\mathfrak{g}_i$ [3, 1.12.10]. Therefore

$$M(f) = M_0(f_0) \supset M_1(f_1) \supset \ldots \supset M_k(f_k), \tag{2}$$

where $M_i(f_i)$ is the $U(\mathfrak{g}_i)$-module induced from the restriction of f_i to $\mathfrak{pv}_i(f_i)$. Note that, as $\mathfrak{g}_k = \{0\}$, then $U(\mathfrak{g}_k) = K$ and $M_k(f_k) = Kl$, where $l = 1 \otimes 1 \in M(f)$.

We denote the module $M(f)$ with the fixed filtration (2) by $M_s(f)$. Note that $M_s(f)$ is an irreducible $U(\mathfrak{g}, s)$-module.

2 Irreducible representations of Lie algebra $(\mathfrak{g}, s)$

The main result of this section is Theorem 2.4. To prove we need several lemmas.

Let $\mathfrak{g}_1$ be an ideal of codimension one in $\mathfrak{g}$. For any $f_1 \in \mathfrak{g}_1^*$ we introduce notation

$$\mathfrak{g}^{f_1} = \{x \in \mathfrak{g} \mid f_1([x, \mathfrak{g}_1]) = 0\}.$$

Lemma 2.1. *Let $f_1 \in \mathfrak{g}_1^*$ and $\mathfrak{g}^{f_1} + \mathfrak{g}_1 = \mathfrak{g}$. Choose $t \in \mathfrak{g}^{f_1}$ and $t \notin \mathfrak{g}_1$. Let $\mathfrak{p}_1$ be a polarization of f_1 in $\mathfrak{g}_1$ that is invariant with respect to ad_t. Suppose that the module M_1, being equal to $\mathrm{ind}(f|_{\mathfrak{p}_1}, \mathfrak{g}_1)$, is irreducible. Then any $U(\mathfrak{g})$-submodule in $\mathrm{ind}(M_1, \mathfrak{g})$ has the form $U(\mathfrak{g})P(t)l$, where $P(t)$ is some polynomial in t and $l = 1 \otimes 1$.*

Proof. There exists the natural filtration of $U(\mathfrak{g}_1)$-submodules in the $U(\mathfrak{g}_1)$-module $\mathrm{ind}(M_1, \mathfrak{g})$:

$$\{0\} = M_1^{(-\infty)} \subset M_1^{(0)} \subset M_1^{(1)} \subset \ldots \subset M_1^{(n)} \subset \ldots, \tag{3}$$

where $M_1^{(n)} = \oplus_{i \leqslant n} t^i M_1$. For any nonzero a in $\mathrm{ind}(M, \mathfrak{g})$, we denote by $\deg(a)$ the smallest n such that $a \in M_1^{(n)}$. If $a = 0$, then we put $\deg(a) = -\infty$. We call $\deg(a)$ the *degree* of a.

Item 1. Let W be a nonzero submodule of $\mathrm{ind}(M_1, \mathfrak{g})$ and let a be an element of smallest degree in W. In this item we shall prove that $W = U(\mathfrak{g})a$.

Let $b \in W$. Using the induction method on $\deg(b)$ we shall show that $b \in U(\mathfrak{g})a$. Case $\deg(b) = -\infty$ (i.e. $b = 0$) is trivial. Suppose that any element in W of degree smaller than m belongs to $U(\mathfrak{g})a$. We shall prove the statement for elements of degree m. By assumption we have $m \geqslant n$.

Let $b = t^m b_0 + \ldots + b_m$, $a = t^n a_0 + \ldots + a_n$, where $a_i, b_j \in M_1$ and $a_0 \neq 0$, $b_0 \neq 0$. Since M_1 is an irreducible $U(\mathfrak{g}_1)$-module, there exists $u \in U(\mathfrak{g}_1)$, such that $b_0 = ua_0$. Then the element $c = t^{m-n}ua - b$ has degree less than m and belongs to W. By induction assumption, $c \in U(\mathfrak{g})a$. Hence $b \in U(\mathfrak{g})a$. This proves $W = U(\mathfrak{g})a$.

Item 2. The element a from item 1 can be presented in the form

$$a = (t^n v_0 + t^{n-1} v_1 + \ldots + v_n)l,$$

where $v_i \in U(\mathfrak{g}_1)$. Since M_1 is an irreducible $U(\mathfrak{g}_1)$-module, there exists $u_0 \in U(\mathfrak{g}_1)$ such that $u_0 v_0 l = l$. The element a', that is equal to $u_0 a$, belongs to W

and generates W as a $U(\mathfrak{g}_1)$-module (indeed, any element of degree n in W generates W) and have the form

$$a' = (t^n + t^{n-1}w_1 + \ldots + w_n)l,$$

where $w_i \in U(\mathfrak{g}_1)$.

Since the polarization $\mathfrak{p}_1$ is invariant with respect to ad_t and $f_1([t, \mathfrak{p}_1]) = 0$, we have

$$(y - f_1(y))t^n l = (t - \mathrm{ad}_t)^n (y - f_1(y))l = 0$$

for every $y \in \mathfrak{p}_1$. Therefore

$$(y - f_1(y))a' = (y - f_1(y))(t^n + t^{n-1}w_1 + \ldots + w_n)l = t^{n-1}(y - f_1(y))w_1 l + a'_1,$$

where $\deg a'_1 < n-1$. Since degree of the element $(y - f_1(y))a'$ is less than n, we have $(y - f_1(y))a' = 0$ and hence $(y - f_1(y))w_1 l = 0$ for any $y \in \mathfrak{p}_1$. There exists a unique endomorphism ϕ of the space M_1, commuting with representation of $\mathfrak{g}_1$ in M_1, and such that $\phi(l) = w_1 l$. Since M_1 is irreducible, the endomorphism ϕ is scalar [3, 2.6.5]. Therefore $w_1 = \alpha_1 \in K$ and $a' = (t^n + t^{n-1}\alpha_1 + t^{n-2}w_2 + \ldots + w_n)l$. Arguing as above one can prove step by step that $w_2, \ldots, w_n$ belong to K. That is $W = U(\mathfrak{g})P(t)l$ for some polynomial $P(t)$ over the field K. □

Corollary 2.2. *Let $\mathfrak{g}_1$, M_1, f_1 be as in lemma 2.1. Any maximal submodule in* $\mathrm{ind}(M_1, \mathfrak{g})$ *has the form $U(\mathfrak{g})(t - \alpha)l$ for some $\alpha \in K$.*

Lemma 2.3. *Let $\mathfrak{g}$, $\mathfrak{g}_1$, f_1 be as in the Lemma 2.1. Denote by π_1 projection $\mathfrak{g}^* \to \mathfrak{g}_1^*$. Let $\mathfrak{p}_1$ be a polarization of f_1 in $\mathfrak{g}_1$. Suppose that $\mathfrak{g}^{f_1} + \mathfrak{g}_1 = \mathfrak{g}$. Then*
1) $\mathfrak{g}^f + \mathfrak{g}_1 = \mathfrak{g}$ for any $f \in \pi_1^{-1}(f_1)$;
2) if $\mathfrak{p}$ is a polarization for some $f \in \pi_1^{-1}(f_1)$ and $\mathfrak{p} \cap \mathfrak{g}_1 = \mathfrak{p}_1$, then $\mathfrak{p} = \mathfrak{g}^{f_1} + \mathfrak{p}_1$.

Proof. 1) It suffices to prove that if $x \in \mathfrak{g}^{f_1}$ and $x \notin \mathfrak{g}_1$, then $x \in \mathfrak{g}^f$. Indeed,

$$f([x, \mathfrak{g}]) = f([x, Kx + \mathfrak{g}_1]) = f([x, \mathfrak{g}_1]) = f_1([x, \mathfrak{g}_1]) = 0.$$

2) Every polarization contains the stabilizer $\mathfrak{g}^f$. Hence $\mathfrak{p} = Kx \oplus \mathfrak{p}_1$, this proves 2). □

Theorem 2.4. *Every irreducible $U(\mathfrak{g}, s)$-module has the form $M_s(f)$ for some $f \in \mathfrak{g}^*$.*

Proof. As we saw in 1.9, $M_s(f)$ is an irreducible $U(\mathfrak{g}, s)$-module. We shall prove the theorem using induction on the length k of filtration. For $k = 0$ the statement is obvious. To conclude the proof it suffices to show that the following statement is true.

Let $\mathfrak{g}_1$ be the first ideal in filtration (1), $f_1 \in \mathfrak{g}_1^*$ and M_1 be an irreducible $U(\mathfrak{g}_1)$-module induced by the character $f_1|_{\mathfrak{pv}_1(f_1)}$ of the Vergne polarization. Let M be an irreducible $U(\mathfrak{g})$-module, that contains M_1 as an $U(\mathfrak{g}_1)$-submodule. We require to prove that $M = M(f)$ for some $f \in \pi^{-1}(f_1)$.

The embedding M_1 into M extends to $U(\mathfrak{g})$-homomorphism $\Psi : \mathrm{ind}(M_1, \mathfrak{g}) \to M$. Since the module M is irreducible, we have $\mathrm{Im}(\Psi) = M$.

Consider two cases a) $\mathfrak{g}^{f_1} \subset \mathfrak{g}_1$, b) $\mathfrak{g}^{f_1} + \mathfrak{g}_1 = \mathfrak{g}$.

a) $\mathfrak{g}^{f_1} \subset \mathfrak{g}_1$. As $\mathfrak{g}^f \subset \mathfrak{g}^{f_1}$, we have $\mathfrak{pv}(f) \subset \mathfrak{g}_1$ for any $f \in \pi^{-1}(f_1)$. The Vergne polarization $\mathfrak{pv}(f)$ coincides with the Vergne polarization of f_1 in $\mathfrak{g}_1$. Therefore $M(f) = \mathrm{ind}(M_1, \mathfrak{g})$ and $\Psi : M(f) \to M$. Both modules $M(f)$ and M are irreducible; the Schur Lemma implies that Ψ is an isomorphism of $M(f)$ onto M.

b) $\mathfrak{g}^{f_1} + \mathfrak{g}_1 = \mathfrak{g}$. The polarization $\mathfrak{p}_1 = \mathfrak{pv}_1(f_1)$ satisfies the conditions of lemma 2.1 (see [3, 1.12.10, 6.1.1]). The kernel of homomorphism Ψ is a maximal submodule in $\mathrm{ind}(M_1, \mathfrak{g})$. By Corollary 2.2, we obtain that the kernel Ψ coincides with $U(\mathfrak{g})(t - \alpha)l$ for some $\alpha \in K$. Let f be an element of $\mathfrak{g}^*$ such that $f(t) = \alpha$ and $\pi_1(f) = f_1$. By Lemma 2.3, $t \in \mathfrak{g}^f$ and hence $\mathfrak{pv}(f) = Kt + \mathfrak{pv}_1(f_1)$. The module $M(f)$ coincides with factor of $\mathrm{ind}(M_1, \mathfrak{g})$ by $\mathrm{Ker}(\Psi)$ and, therefore, Ψ isomorphically maps $M(f)$ onto M. □

3 Mapping M_s

In this section we shall answer a question when two modules $M_s(f')$ and $M_s(f'')$ are equivalent.

Lemma 3.1. *There exists a unique nonzero (up to scalar multiple) element l in $M(f)$ such that*

$$(x - f(x))l = 0. \tag{4}$$

Proof. This element exists, since the element $l = 1 \otimes 1$ satisfies this property. On the other hand, if $\tilde{l}$ satisfies (4), then there exists commuting with representation endomorphism ϕ such that $\phi(l) = \tilde{l}$. As $M(f)$ is an irreducible $U(\mathfrak{g})$-module, the endomorphism ϕ is scalar [3, 2.6.5]. □

Lemma 3.2. *If an element l satisfies (4), then the equality $(y - \lambda)l = 0$ implies $y \in \mathfrak{pv}(f)$ and $\lambda = f(y)$, where $y \in \mathfrak{g}$ и $\lambda \in K$.*

Proof. By the Poincaré-Birkhoff-Witt theorem (see [9, 2.7]), the system

$$\left\{x_1^{k_1} \cdots x_n^{k_n} \ : \ k_1, \ldots, k_n \in \mathbb{Z}_+\right\}$$

is a basis of $U(\mathfrak{g})$ for any basis $x_1, \ldots, x_n$ of $\mathfrak{g}$. Let $x_1, \ldots, x_m, x_{m+1}, \ldots, x_n$ be a basis of $\mathfrak{g}$ such that $x_{m+1}, \ldots, x_n$ is a basis of $\mathfrak{pv}(f)$. Then

$$\left\{x_1^{k_1} \cdots x_m^{k_m} l \ : \ k_1, \ldots, k_m \in \mathbb{Z}_+\right\}$$

is a basis of $M_s(f)$. $\square$

Lemma 3.3. *Let $f \in \mathfrak{g}^*$ and f_1 be a restriction of f on the ideal $\mathfrak{g}_1$ of codimension one from (1). Then*
a) if $\mathfrak{g}^f + \mathfrak{g}_1 = \mathfrak{g}$, then $M(f)$ is isomorphic to $M_1(f_1)$ as a $U(\mathfrak{g}_1)$-module;
b) if $\mathfrak{g}^f \subset \mathfrak{g}_1$, then $M(f)$ admits an infinite filtration, where each factor is isomorphic to $M_1(f_1)$.

Proof. The statement a) is obvious. Suppose that the assumption of statement b) holds. Choose $t \in \mathfrak{g} \setminus \mathfrak{g}_1$. As in proof of Lemma 2.1, consider filtration (3) with $M_1 = M_1(f_1)$.

Consider the map $\phi : M_1 \to M_1^{(n)}$ such that $\phi(v) = t^n v, \ \ v \in M_1$. For any $u_1 \in U(\mathfrak{g}_1)$ we have $u_1\phi(v) = u_1 t^n v = t^n u_1 v \mod M_1^{(n-1)}$. The map ϕ is a homomorphism of $U(\mathfrak{g}_1)$-modules. It can de extended to an isomorphism of M_1 to the factor of $M_1^{(n)}$ by $M_1^{(n-1)}$. $\square$

Theorem 3.4. *The following conditions are equivalent:*
1) $M_s(f')$ and $M_s(f'')$ are equivalent as $U(\mathfrak{g})$-modules;
2) $M_s(f')$ and $M_s(f'')$ are equivalent as $U(\mathfrak{g}, s)$-modules;
3) the Vergne polarizations of f' and f'' coincide, and f' equals to f'' under restriction to a common Vergne polarization.

Proof. Implications 3) $\Rightarrow$ 2) $\Rightarrow$ 1) is obvious. Let us prove 1) $\Rightarrow$ 3) by induction on k. For $k = 1$ the statement if easy. Suppose that the statement is proved for filtrations of length less than k. Let us prove for k. Denote by s_1 the restriction of the filtration s to $\mathfrak{g}_1$.

Assume that $M_s(f')$ and $M_s(f'')$ are equivalent as $U(\mathfrak{g})$-modules. Then they are equivalent as $U(\mathfrak{g}_1)$-modules. By Lemma 3.3, f' and f'' satisfy simultaneously either condition a), or b) of this Lemma, and $U(\mathfrak{g}_1)$-modules $M_{s_1}(f_1')$ and $M_{s_1}(f_1'')$ are isomorphic, where f_1' and f_1'' are restrictions of f' and f'' to $\mathfrak{g}_1$. Using the the inductive assumption, we obtain:
i) $\mathfrak{pv}_1(f_1') = \mathfrak{pv}_1(f_1'')$ (denote it by $\mathfrak{p}_1$),
ii) $f'|_{\mathfrak{p}_1} = f''|_{\mathfrak{p}_1}$.

If f' and f'' satisfy condition b) of Lemma 3.3, then $\mathfrak{p}_1$ is their common Vergne polarization; and ii) proves 3).

If f' and f'' satisfy condition a), then an isomorphism $\phi : M_s(f') \to M_s(f'')$ as $U(\mathfrak{g})$-modules is an isomorphism $M_{s_1}(f_1') \to M_{s_1}(f_1'')$ of $U(\mathfrak{g}_1)$-modules. The conditions i) and ii) imply that $M_{s_1}(f_1') = M_{s_1}(f_1'') = \mathrm{ind}(f_1'|_{\mathfrak{p}_1}, \mathfrak{g}_1)$. Since ϕ is an isomorphism of irreducible module, the operator ϕ is scalar. Therefore $M_s(f')$ coincides with $M_s(f'')$ as a $U(\mathfrak{g})$-module.

Let l be an element of this module, satisfying (4). By $M_s(f') = M_s(f'')$, the condition (4) holds for both cases $f = f', \ x \in \mathfrak{pv}(f')$ and $f = f''$, $x \in \mathfrak{pv}(f'')$.

The Lemma 3.2 implies that $\mathfrak{pv}(f') = \mathfrak{pv}(f'')$ and f' coincides with f'' under restriction to a common Vergne polarization. $\square$

Definition 3.5. Consider the equivalence relation $\mathfrak{R}$ on $\mathfrak{g}^*$ such that $f'\mathfrak{R}f''$ if f' and f'' have a common Vergne polarization and coincide under restriction to it. We denote the equivalence class of an arbitrary element $f \in \mathfrak{g}^*$ by $\mathfrak{R}(f)$.

By theorem 3.4, the correspondence $M_s : f \mapsto M_s(f)$ extends to bijection $\mathfrak{g}^*/\mathfrak{R}$ to $\mathrm{Irr}(\mathfrak{g}, s)$. Our next goal is to describe equivalence classes for $\mathfrak{R}$.

Lemma 3.6. *Let $\mathfrak{g}_1$ be an ideal of codimension one in $\mathfrak{g}$, $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is some polarization of $f \in \mathfrak{g}^*$ (resp. $g \in \mathfrak{g}^*$) such that $\mathfrak{p} \cap \mathfrak{g}_1 = \mathfrak{p}' \cap \mathfrak{g}_1 = \mathfrak{p}_1$, where $\mathfrak{p}_1$ is a common polarization of projections f_1 and g_1 for elements f and g to $\mathfrak{g}_1^*$. Let $f|_{\mathfrak{p}_1} = g|_{\mathfrak{p}_1}$. Then $\mathfrak{p} = \mathfrak{p}'$.*

Proof. The polarizations $\mathfrak{p}$ and $\mathfrak{p}'$ can be written in the form $\mathfrak{p} = Kx + \mathfrak{p}_1$ and $\mathfrak{p}' = Ky + \mathfrak{p}_1$ for some $x, y \in \mathfrak{g}$. Since $f|_{\mathfrak{p}_1} = g|_{\mathfrak{p}_1}$, we have

$$f([y, \mathfrak{p}_1]) = g([y, \mathfrak{p}_1]) = 0.$$

Therefore, $\mathfrak{p}'$ is an isotropic subspace not only for g, but for f too. Similarly, $\mathfrak{p}$ is an isotropic subspace for g. In particular, the case $\mathfrak{p} = \mathfrak{p}_1 \neq \mathfrak{p}'$ as well as $\mathfrak{p}' = \mathfrak{p}_1 \neq \mathfrak{p}$ is not possible.

Hence either $\mathfrak{p} = \mathfrak{p}' = \mathfrak{p}_1$ (that implies the statement of the lemma), or $\mathfrak{p} \neq \mathfrak{p}_1$ and $\mathfrak{p}' \neq \mathfrak{p}_1$. In the second case, $x, y \notin \mathfrak{g}_1$ and one can choose x, y such that $x - y \in \mathfrak{g}_1$. Since

$$f_1([x - y, \mathfrak{p}_1]) = f([x - y, \mathfrak{p}_1]) = f([x, \mathfrak{p}_1]) - f([y, \mathfrak{p}_1]) = 0,$$

the space $K(x - y) + \mathfrak{p}_1$ is isotropic for f_1 in $\mathfrak{g}_1$. As $\mathfrak{p}_1$ is a polarization of f_1, we have $x - y \in \mathfrak{p}_1$ and, finally, $\mathfrak{p} = \mathfrak{p}'$. $\square$

Theorem 3.7. *The equivalence class $\mathfrak{R}(f)$ of an element $f \in \mathfrak{g}$ coincides with $\pi^{-1}\pi(f)$, where π is projection $\mathfrak{g}^*$ to $\mathfrak{pv}(f)^*$.*

Proof. By theorem 3.4, $\mathfrak{R}(f) \subset \pi^{-1}\pi(f)$. We have to show that $\mathfrak{pv}(g) = \mathfrak{pv}(f)$ for every $g \in \pi^{-1}\pi(f)$. Using induction on the length of filtration, we obtain $\mathfrak{pv}_1(f_1) = \mathfrak{pv}_1(g_1)$. The Lemma 3.6 implies $\mathfrak{pv}(f) = \mathfrak{pv}(g)$. $\square$

4 Spectrum of certain $(\mathfrak{g}, s)$-modules

Let $\mathfrak{h}$ be a subalgebra of Lie algebra $\mathfrak{g}$. A filtration s on $\mathfrak{g}$ induces a filtration on $\mathfrak{h}$ and turns it into a Lie algebra $(\mathfrak{h}, s)$ with filtration. Let h be an element in $\mathfrak{h}^*$. Denote by $\mathfrak{qv}(h)$ the Vergne polarization of h in $(\mathfrak{h}, s)$ and by $N_s(h)$ – the representation $\mathrm{ind}(h|_{\mathfrak{qv}(h)}, \mathfrak{h})$. Since $N_s(h)$ is a representation of the Lie algebra $(\mathfrak{h}, s)$, then $\mathrm{ind}(N_s, \mathfrak{g})$ is a representation of $(\mathfrak{g}, s)$ (see 1.7). The goal of this section is to describe spectra of the induced representation $\mathrm{ind}(N_s, \mathfrak{g})$, restriction $M_s(f)|_{(\mathfrak{h},s)}$ and of representation $M_s(f') \otimes M_s(f'')$.

Lemma 4.1.*Let Z_s be a submodule of* $\mathrm{ind}(N_s(h),\mathfrak{g})$*; the factor module by Z_s is isomorphic to $M_s(f)$ as a $U(\mathfrak{g},s)$-module. Then*
1) $\mathfrak{pv}(f)\supset\mathfrak{qv}(h)$*;*
2) $f|_{\mathfrak{qv}(h)}=h|_{\mathfrak{qv}(h)}$*;*
3) $Z_s=\sum_{x\in\mathfrak{pv}(f)}U(\mathfrak{g})(x-f(x))l_h$*, where l_h is an element* $1\otimes 1$.

Proof. Use the induction on the filtration length k. For $k=0$ the statement is obvious. Assume that the statement is true for length less than k. Let us prove it for the length k. Let $\mathfrak{g}_1$ be an ideal of codimension one in filtration (1). Introduce the notations: $\mathfrak{h}_1=\mathfrak{h}\cap\mathfrak{g}_1$, h_1 (resp. f_1) – projection of h (resp. f) to $\mathfrak{h}_1$ (resp. $\mathfrak{g}_1$), $V_s=\mathrm{ind}(N_s(h),\mathfrak{g})$, s_1 – restriction of filtration s to $\mathfrak{g}_1$, $V_{s_1}=\mathrm{ind}(N_{s_1}(h_1),\mathfrak{g}_1)$, $Z_{s_1}=Z_s\cap V_{s_1}$.

By assumption, $V_s/Z_s\cong M_s(f)$. Hence $V_{s_1}/Z_{s_1}\cong M_{s_1}(f_1)$. By the inductive assumption:
1') $\mathfrak{pv}(f_1)\supset\mathfrak{qv}(h_1)$;
2') $f_1|_{\mathfrak{qv}(h_1)}=h_1|_{\mathfrak{qv}(h_1)}$;
3') $Z_{s_1}=\sum_{x\in\mathfrak{pv}(f_1)}U(\mathfrak{g})(x-f_1(x))l_{h_1}$, where $l_{h_1}=l_h=1\otimes 1$. Consider two cases: a) $\mathfrak{qv}(h)+\mathfrak{g}_1=\mathfrak{g}$ и b) $\mathfrak{qv}(h)\subset\mathfrak{g}_1$.
a) $\mathfrak{qv}(h)+\mathfrak{g}_1=\mathfrak{g}$. In this case $V_s=V_{s_1}$ and, therefore, $Z_s=Z_{s_1}$, that is Z_{s_1} is a $U(\mathfrak{g})$-submodule.

Let $x\in\mathfrak{pv}_1(f_1)$. It follows from 3') that $(x-f_1(x))l_h\in Z_{s_1}$. Since Z_{s_1} is a $U(\mathfrak{g})$-module, $y(x-f_1(x))l_h\in Z_{s_1}$ for every $y\in\mathfrak{g}$. Choose $y\in\mathfrak{qv}(h)$ and $y\notin\mathfrak{g}_1$. The following equality

$$y(x-f_1(x))l_h=(x-f_1(x))yl_h+[y,x]l_h=h(y)(x-f_1(x))l_h+[y,x]l_h$$

implies $[y,x]l_h\in Z_{s_1}$. After factorization by Z_{s_1} we obtain that $[y,x]l=0$ holds in $M_{s_1}(f_1)$,where $l=1\otimes 1$. By lemma 3.2, we have got $[y,\mathfrak{pv}_1(f_1)]\subset\mathfrak{pv}_1(f_1)$ and $f_1([y,\mathfrak{pv}_1(f_1)])=0$.

So, the subalgebra $\mathfrak{p}$, defined as $Ky+\mathfrak{pv}(f_1)$, is a polarization for every $g\in\pi_1^{-1}(f_1)$, where π_1 is projection of $\mathfrak{g}^*$ to $\mathfrak{g}_1^*$. Choose $g\in\pi_1^{-1}(f_1)$ such that $g(y)=h(y)$. Using inductive assumption, we obtain
1") $\mathfrak{p}\supset\mathfrak{qv}(h)$;
2") $g|_{\mathfrak{qv}(h)}=h|_{\mathfrak{qv}(h)}$.
Since $y\in\mathfrak{qv}(h)$, we have $(y-g(y))l_h=(y-h(y))l_h=0$. Recall that in our case $Z_s=Z_{s_1}$. Applying 3'), we obtain

$$3")\quad Z_{s_1}=\sum_{x\in\mathfrak{pv}_1(f_1)}U(\mathfrak{g})(x-f_1(x))l_h=\sum_{x\in\mathfrak{p}}U(\mathfrak{g})(x-g(x))l_h.$$

Since $\mathfrak{p}\cap\mathfrak{g}_1=\mathfrak{pv}_1(f_1)$ and $\mathfrak{pv}(g)\cap\mathfrak{g}_1=\mathfrak{pv}_1(f_1)$, we have $\mathfrak{p}=\mathfrak{pv}(g)$ (see Lemma 3.6). As $V_s=\mathrm{ind}(N_s(h),\mathfrak{g})=\mathrm{ind}(h|_{\mathfrak{qv}(h)},\mathfrak{g})$ and Z_s has the form 3"), we conclude $V_s/Z_s\cong M_s(g)$. By assumption, we have $V_s/Z_s\cong M_s(f)$.

Therefore, $M_s(f) \cong M_s(g)$. Using theorem 3.4, we obtain $\mathfrak{pv}(g) = \mathfrak{pv}(f) = \mathfrak{p}$ and $f|_{\mathfrak{p}} = g|_{\mathfrak{p}}$. Substituting this equalities in 1"), 2"), 3"), we get 1), 2), 3). This proves lemma in the case a).
b) $\mathfrak{qv}(h) \subset \mathfrak{g}_1$. In this case, $V_s = \mathrm{ind}(V_{s_1}, \mathfrak{g})$, $\quad Z_s \supset \mathrm{ind}(Z_{s_1}, \mathfrak{g})$ and

$$V_s/\mathrm{ind}(Z_{s_1}, \mathfrak{g}) = \mathrm{ind}(V_{s_1}, \mathfrak{g})/\mathrm{ind}(Z_{s_1}, \mathfrak{g}) = \mathrm{ind}(M_{s_1}(f_1), \mathfrak{g}).$$

Denote by $\overline{Z}_s$ the image of Z_s in factor of V_s with respect to $\mathrm{ind}(Z_{s_1}, \mathfrak{g})$. Consider two cases.
b1) $\mathfrak{g}^{f_1} \subset \mathfrak{g}_1$. In this case, $\mathfrak{pv}(f) = \mathfrak{pv}_1(f_1)$ (this proves 1) and 2)) and $U(\mathfrak{g})$-module $\mathrm{ind}(M_{s_1}(f_1), \mathfrak{g})$ is irreducible. Hence, $\overline{Z}_s = 0$ and, therefore,

$$Z_s = \mathrm{ind}(Z_{s_1}, \mathfrak{g}) = \sum_{x \in \mathfrak{pv}(f) = \mathfrak{pv}_1(f_1)} U(\mathfrak{g})(x - f_1(x))l_h;$$

this proves 3).
b2) $\mathfrak{g}^{f_1} + \mathfrak{g}_1 = \mathfrak{g}$. In this case, the Vergne polarization $\mathfrak{pv}(f)$ has the form $Ky \oplus \mathfrak{pv}_1(f_1)$, where $y \in \mathfrak{pv}(f) \setminus \mathfrak{g}_1$ (see lemma 3.6). By corollary 2.2, we get $\overline{Z}_s = U(\mathfrak{g})(y - f(y))l_h$, that implies the statements 1), 2) and 3). □

Corollary 4.2. *The $U(\mathfrak{g}, s)$-module $M_s(f)$ is isomorphic to factor of* $\mathrm{ind}(N_s(h), \mathfrak{g})$ *if and only if*
1) $\mathfrak{pv}(f) \subset \mathfrak{qv}(h)$ and 2) $f|_{\mathfrak{qv}(h)} = h|_{\mathfrak{qv}(h)}$.

Proof. It is immediate from Lemma 4.1. □

Theorem 4.3. *Let $\mathfrak{r}(h)$ be the equivalence class of $h \in \mathfrak{h}^*$, π be the projection $\mathfrak{g}^*$ на $\mathfrak{h}^*$. Then*

$$\mathrm{Spec}(\mathrm{ind}(N_s(h), \mathfrak{g})) = \left\{ M_s(f) \mid \mathfrak{R}(f) \subset \pi^{-1}(\mathfrak{r}(h)) \right\}.$$

Here, as above, $\mathfrak{R}(f)$ is the equivalence class of $f \in \mathfrak{g}^*$.

Proof. The $U(\mathfrak{g}, s)$-module $V_s(h)$ equals to $\mathrm{ind}(N_s(h), \mathfrak{g})$ and has the filtration

$$V_0(f_0) \supset V_1(f_1) \supset \ldots \supset V_k(f_k),$$

here $V_k(h_k) = kl$, $\quad l = 1 \otimes 1$.

Suppose that $M_s(f) \in \mathrm{Spec}(V_s(h))$. This means that there exist $U(\mathfrak{g}, s)$-submodules W_s and W'_s such that $V_s \supset W_s \supset W'_s$ and W_s/W'_s is isomorphic to $M_s(f)$. This is equivalent to the existence of chains $W_0 \supset W_1 \supset \ldots \supset W_k$ and $W'_0 \supset W'_1 \supset \ldots \supset W'_k$ such that W_0, W'_0 are $U(\mathfrak{g})$-modules, $W_i = W_0 \cap V_i(h_i)$, $W'_i = W'_0 \cap V_i(h_i)$, $V_i(h_i) \supset W_i \supset W'_i$ and W_i/W'_i is isomorphic to $M_i(f_i)$ as $U(\mathfrak{g}_i)$-module. For $i = k$ we have $V_k(h_k) \supset W_k \supset W'_k$ and W_k/W'_k is isomorphic to $M_i(f_i)$ as a linear space over K. Since $\dim V_k(h_k) = \dim M_i(f_i) = 1$, we conclude that $W'_k = \{0\}$ and $W_k = V_k(h_k) = Kl$. Hence $l \in W_s$. By $V_s(h) = U(\mathfrak{g})l$, $W_s = V_s$ and any $U(\mathfrak{g}, s)$-subfactor module is a factor module.

This shows that $M_s(f) \in \mathrm{Spec}\,(V_s(h))$ if and only if f and h satisfy the conditions 1) and 2) of Corollary 4.2.

We have

$$\mathfrak{R}(f) = \left\{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{pv}(f)} = f|_{\mathfrak{pv}(f)}\right\},$$
$$\pi^{-1}(\mathfrak{r}(h)) = \left\{\eta \in \mathfrak{g}^* \mid \eta|_{\mathfrak{qv}(h)} = h|_{\mathfrak{qv}(h)}\right\}$$

and conditions 1) and 2) of Corollary 4.2 are equivalent to $\mathfrak{R}(f) \subset \pi^{-1}(\mathfrak{r}(h))$. $\square$

Theorem 4.4.*Let* $\mathfrak{r}(h)$, $\mathfrak{R}(f)$, π *be as in previous theorem. Then*

$$\mathrm{Spec}\left(M_s(f)|_{(\mathfrak{h},s)}\right) = \left\{N_s(h) \mid \mathfrak{r}(h) \subset \pi(\mathfrak{R}(f))\right\}.$$

Proof. Suppose that $N_s(h) \in \mathrm{Spec}\, M_s(f)|_{(\mathfrak{h},s)}$. This means that there exist $U(\mathfrak{h},s)$-submodules C_s and C'_s such that $C'_s \supset C_s$ and C_s/C'_s is isomorphic to $N_s(h)$. Arguing as in the proof of previous theorem, we obtain $C'_k \supset C_k \supset M_k(f_k)$ and $C_k/C'_k \cong N_k(h_k)$. Since $\dim M_k(f_k) = \dim N_k(h_k) = 1$, we have $C'_k = \{0\}$ and $C_k = M_k(f_k) = Kl$, where $l = 1 \otimes 1$. Hence $l \in C_k$ and $U(\mathfrak{h})l \subset C_s$. This implies that $N_s(h)$ is a factor of $U(\mathfrak{h},s)$-module $U(\mathfrak{h})l$, that is isomorphic to $\mathrm{ind}(f|_{\mathfrak{pv}(f)\cap\mathfrak{h}}, \mathfrak{h})$. Applying Corollary 4.2, we obtain that $N_s(h)$ is contained in $\mathrm{Spec}\left(M_s(f)|_{(\mathfrak{h},s)}\right)$ if and only if

$$\begin{cases} \mathfrak{qv}(h) \supset \mathfrak{pv}(f) \cap \mathfrak{h}, \\ h|_{\mathfrak{pv}(f)\cap\mathfrak{h}} = f|_{\mathfrak{pv}(f)\cap\mathfrak{h}}. \end{cases} \tag{5}$$

Since

$$\mathfrak{r}(h) = \left\{\lambda \in \mathfrak{h}^* \mid \lambda|_{\mathfrak{qv}(h)} = h|_{\mathfrak{qv}(h)}\right\},$$
$$\pi(\mathfrak{R}(f)) = \left\{\mu \in \mathfrak{h}^* \mid \mu|_{\mathfrak{pv}(f)\cap\mathfrak{h}} = f|_{\mathfrak{pv}(f)\cap\mathfrak{h}}\right\},$$

tha conditions (5) are equivalent to $\mathfrak{r}(h) \subset \pi(\mathfrak{R}(f))$.

$\square$

Let $f', f'' \in \mathfrak{g}^*$. The Vergne polarization of $f' \times f'' \in (\mathfrak{g} \times \mathfrak{g})^*$ with respect to filtration $s \times s$ (see 1.8) coincides with $\mathfrak{pv}(f') \times \mathfrak{pv}(f'')$ and

$$M_s(f') \times M_s(f'') = M_{s\times s}(f' \times f'').$$

Theorem 4.5. *Let* $f', f'' \in \mathfrak{g}^*$. *we claim that*

$$\mathrm{Spec}\, M_s(f') \otimes M_s(f'') = \left\{M_s(g) \mid \mathfrak{R}(g) \subset \mathfrak{R}(f') + \mathfrak{R}(f'')\right\}.$$

Proof. By definition, $M_s(f') \otimes M_s(f'')$ is a restriction of $M_s(f') \times M_s(f'')$ to $(\mathfrak{g}, s)$. By Theorem 4.4,

$$\mathrm{Spec}\, M_s(f') \otimes M_s(f'') = \left\{M_s(g) \mid \mathfrak{R}(g) \subset \pi\left(\mathfrak{R}(f' \times f'')\right)\right\},$$

where π is projection $(\mathfrak{g} \times \mathfrak{g})^*$ to $\mathfrak{g}^*$, $\mathfrak{R}(f' \times f'')$ is the equivalence class of $f' \times f''$ in $(\mathfrak{g} \times \mathfrak{g})^*$. Easy to see that

$$\pi\left(\mathfrak{R}(f' \times f'')\right) = \pi\left(\mathfrak{R}(f') \times \mathfrak{R}(f'')\right) = \mathfrak{R}(f') + \mathfrak{R}(f'').\Box$$

5 Connection with Dixmier mapping. Examples

The orbit method of A.A.Kirillov describes irreducible representations of solvable Lie groups in Hilbert spaces. It has an algebraic analogue. Let $\mathfrak{g}$ be a solvable Lie algebra over algebraically closed field K of zero characteristic and let $\mathcal{A}$ be an adjoint algebraic group for $\mathfrak{g}$. Let f be an element of $\mathfrak{g}^*$ and $\mathfrak{p}$ be a polarization of f. We denote by $\theta_{\mathfrak{p}}$ the character of Lie subalgebra $\mathfrak{p}$ given as $\frac{1}{2}\mathrm{trad}_{\mathfrak{g}/\mathfrak{p}}$. The ideal $I(f)$ defined as a kernel of the twisted induced representation $\mathrm{ind\tilde{}}(f|_{\mathfrak{p}}, \mathfrak{g}) = \mathrm{ind}(f - \theta_{\mathfrak{p}}|_{\mathfrak{p}}, \mathfrak{g})$ in the universal enveloping algebra is primitive and does not depend on choice of polarization $\mathfrak{p}$. The correspondence $f \mapsto I(f)$ can be extended to bijection of the orbit space $\mathfrak{g}^*/\mathcal{A}$ to that set $\mathrm{Prim}\, U(\mathfrak{g})$ of primitive ideals of $U(\mathfrak{g})$. This bijection if denoted by $\mathrm{Dix}_{\mathfrak{g}}$ and is called the Dixmier map; see [3, Глава 6].

Theorem 5.1. *1) $\mathfrak{R}(f) \subset \mathcal{A}(f)$ for every $f \in \mathfrak{g}^*$. Denote by $i : \mathfrak{g}^*/\mathfrak{R} \to \mathfrak{g}^*/\mathcal{A}$ the embedding of $\mathfrak{R}(f)$ into $\mathcal{A}(f)$.*
2) Denote by $\mathfrak{Ker} : \mathrm{Irr}(\mathfrak{g}, s) \to \mathrm{Prim}\ U(\mathfrak{g})$ the map sending the irreducible representation $\mathrm{ind}(f|_{\mathfrak{p}}, \mathfrak{g})$ to the kernel of twisted representation $\mathrm{ind\tilde{}}(f|_{\mathfrak{p}}, \mathfrak{g})$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g}^*/\mathfrak{R} & \xrightarrow{i} & \mathfrak{g}^*/\mathcal{A} \\ \big\downarrow{\scriptstyle M_s} & & \big\downarrow{\scriptstyle \mathrm{Dix}_{\mathfrak{g}}} \\ \mathrm{Irr}(\mathfrak{g}, s) & \xrightarrow{\mathfrak{Ker}} & \mathrm{Prim}\, U(\mathfrak{g}) \end{array}$$

Proof. 1) For any $g \in \mathfrak{R}(f)$ we have $\mathfrak{pv}(g) = \mathfrak{pv}(f) = \mathfrak{p}$ and $g|_{\mathfrak{p}} = f|_{\mathfrak{p}}$ (theorem 3.7). Hence the map $g \mapsto I(g) = \mathrm{Ker}\left(\mathrm{ind\tilde{}}(f|_{\mathfrak{p}}, \mathfrak{g})\right)$ is constant on $\mathfrak{R}(f)$. Therefore $\mathfrak{R}(f) \subset \mathcal{A}(f)$.

The statement of item 2) is a corollary of definition of the Dixmier mapping. $\Box$

Example 1. Let $\mathfrak{g} = \mathrm{span} < x, y \mid [x, y] = y >$. Fix the filtration s of the form $\mathfrak{g} \supset Ky \supset \{0\}$.

For $f \in \mathfrak{g}^*$ denote $f(y) = y^0$. If $y^0 \neq 0$, then $\mathfrak{R}(f) = \{g \in \mathfrak{g}^* \mid g(y) = y^0\}$; and if $y^0 = 0$, then $\mathfrak{R}(f) = \{f\}$.

If $y^0 \neq 0$, then the corresponding irreducible representation of $(\mathfrak{g}, s)$ is induced by the character $y \mapsto y^0$ of subalgebra $\mathfrak{p} = Ky$. In the case $y_0 = 0$ linear form f is a character and defines one-dimensional representation of $(\mathfrak{g}, s)$.

Note that a classification of all irreducible representations of Lie algebra $\mathfrak{g}$ are given in [2].

Example 2. Let $\mathfrak{g}$ be the Heisenberg algebra span $< x, y, z \mid [x, y] = z >$. Fix the filtration s of the form $\mathfrak{g} \supset Ky \oplus Kz \supset Kz \supset \{0\}$.

Let $f \in \mathfrak{g}^*$, denote $f(y) = y^0$ and $f(z) = z^0$. If $z^0 \neq 0$, then $\mathfrak{R}(f) = \{g \in \mathfrak{g}^* \mid g(z) = y^0, \ \ g(z) = z^0\}$; and if $z^0 = 0$, then $\mathfrak{R}(f) = \{f\}$.

If $z^0 \neq 0$, then the corresponding irreducible representation of $(\mathfrak{g}, s)$ is induced by the character $y, z \mapsto y^0, z^0$ of the subalgebra $\mathfrak{p} = Ky \oplus Kz$. In the case $z_0 = 0$ the linear form f is a character. It defines one-dimensional representation of $(\mathfrak{g}, s)$.

Concerning a classification of all irreducible representations of the Heisenberg algebra, it is known that every irreducible is either one-dimensional (i.e. coincides with f with $z^0 = 0$), or is an irreducible representation of the Weyl algebra $A_1 =< p, q \mid [p, q] = 1 >$ (see [1]).

Note that in each of these examples irreducible representations of $(\mathfrak{g}, s)$ cover a small part of all irreducible representations of $\mathfrak{g}$ (see [2, Prop. 6.1]).

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A.N.Panov)
Samara State University,
E-mail: apanov@list.ru